Lecture 19
Probabilistic symbolic model checking

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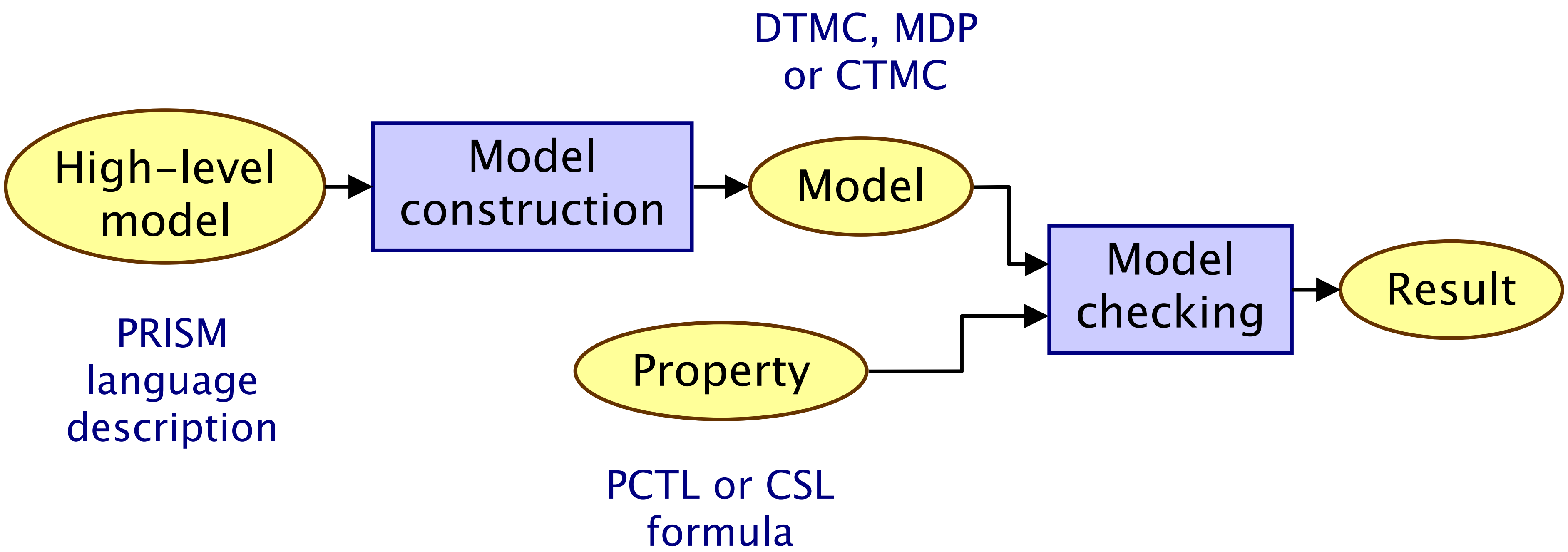
Overview

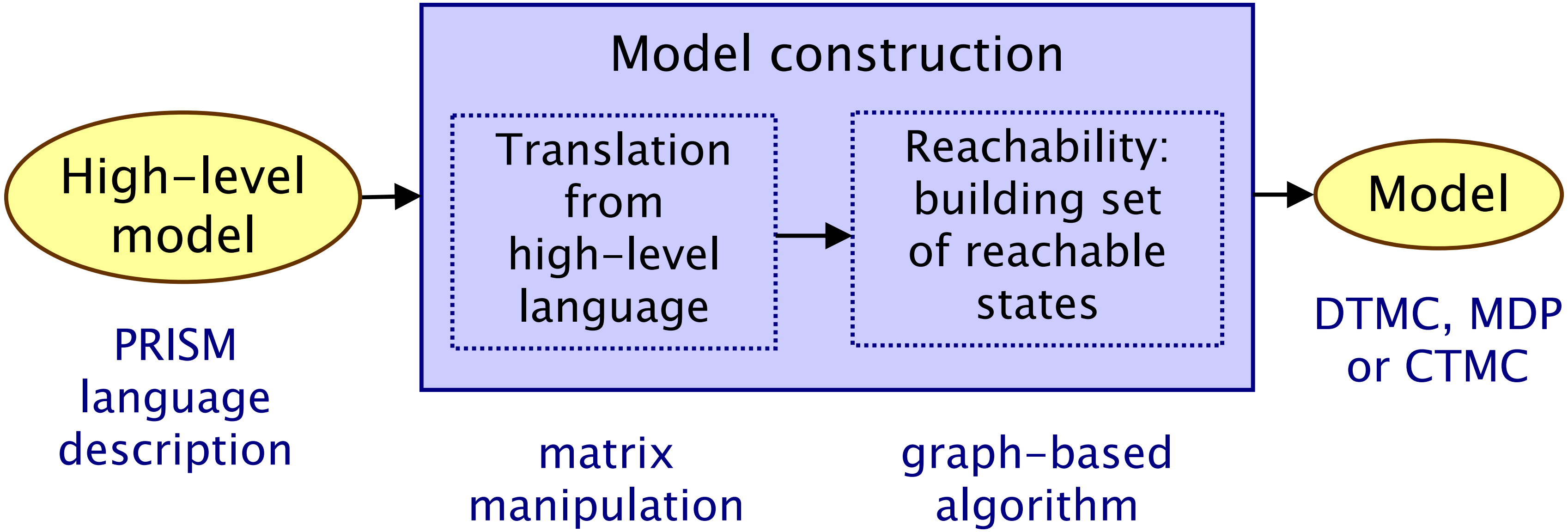
- Implementation of probabilistic model checking
  - overview, key operations, symbolic vs. explicit
- Binary decision diagrams (BDDs)
  - introduction, sets, transition relations, …
- Multi-terminal BDDs (MTBDDs)
  - introduction, vectors, matrices, …
- Operations on/with BDDs and MTBDDs

Implementation overview

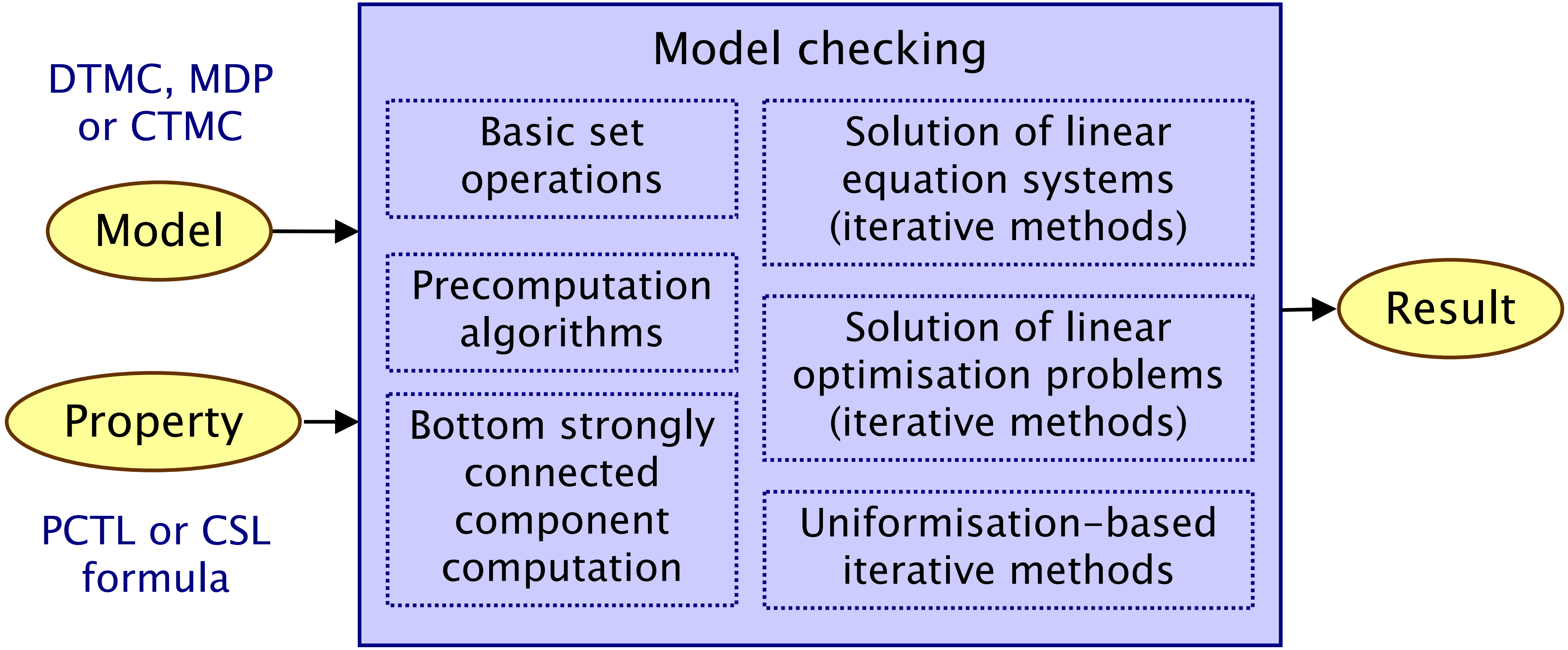
- Overview of the probabilistic model checking process
  - two distinct phases: model construction, model checking
  - three different models, several different logics, various different solution/analysis methods
  - but… all these processes have much in common



Model construction



Model checking



Two distinct classes of techniques:
graph-based algorithms
iterative numerical computation

Underlying operations

- Key objects/operations for probabilistic model checking

- Graph-based algorithms
  - underlying transition relation of DTMC/MDP/CTMC
  - manipulation of transition relation and state sets

- Iterative numerical computation
  - transition matrix of DTMC/MDP/CTMC, real-valued vectors
  - manipulation of real-valued matrices and vectors
  - in particular: matrix-vector multiplication

State-space explosion

- Models of real-life systems are typically huge
  - familiar problem for verification/model checking techniques
- State-space explosion problem
  - linear increase in size of system can result in an exponential increase in the size of the model
  - e.g. n parallel components of size m, can give up to m^n states
- Need efficient ways of storing models, sets of states, etc.
  - and efficient ways of constructing, manipulating them
- Here, we will focus on symbolic approaches

Symbolic data structures

- Distinguish between explicit and symbolic storage
- Symbolic data structures
  - usually based on binary decision diagrams (BDDs) or variants
  - avoid explicit enumeration of data by exploiting regularity
  - potentially very compact storage (but not always)
- Sets of states:
  - explicit: bit vectors, symbolic: BDDs
- Real-valued vectors:
  - explicit: arrays of reals (in practice, doubles/floats)
  - symbolic: multi-terminal BDDs (MTBDDs)
- Real-valued matrices:
  - explicit: sparse matrices
  - symbolic: MTBDDs

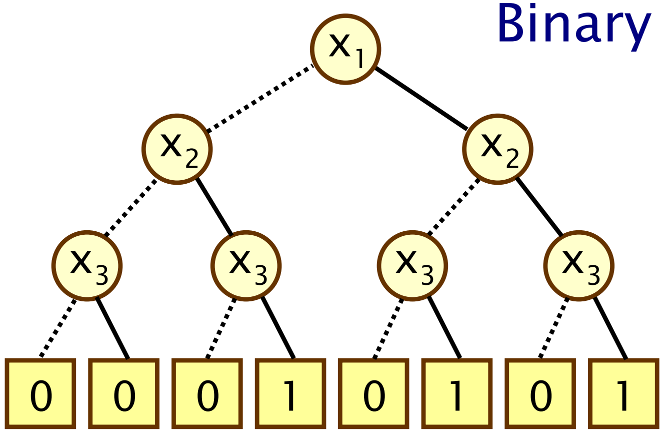
Representations of Boolean formulas

- Propositional formula: $f = (x_1 \vee x_2) \wedge x_3$

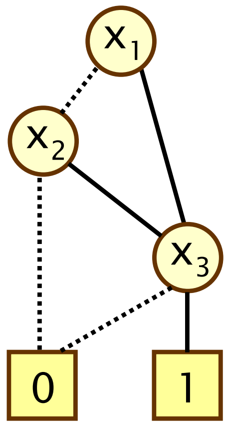
Truth table

x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Binary decision tree

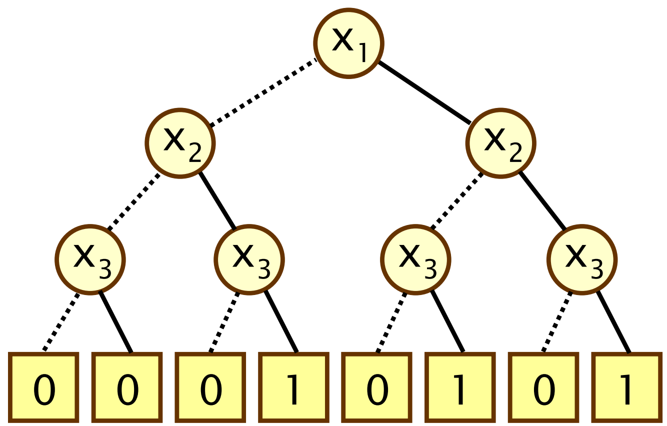


Binary decision diagram



Binary decision trees

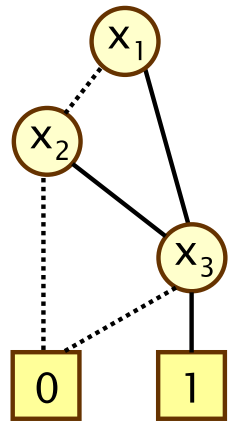
- Graphical representation of Boolean functions
  - $f(x_1,\ldots,x_n) : \{0,1\}^n \rightarrow \{0,1\}$
- Binary tree with two types of nodes
- Non-terminal nodes
  - labelled with a Boolean variable x_i
  - two children: 1 (“then”, solid line) and 0 (“else”, dotted line)
- Terminal nodes (or “leaf” nodes)
  - labelled with 0 or 1
- To read the value of $f(x_1,\ldots,x_n)$
  - start at root (top) node
  - take “then” edge if $x_i=1$
  - take “else” edge if $x_i=0$
  - result given by leaf node



Binary decision diagrams

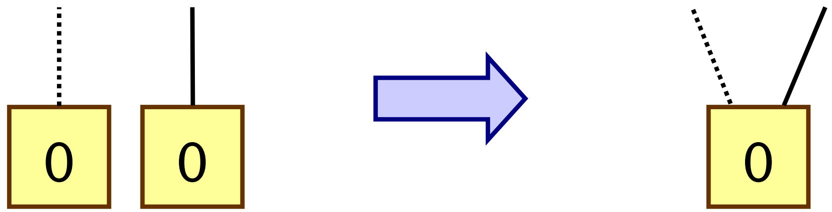
- Binary decision diagrams (BDDs) [Bry86]
  - based on binary decision trees, but reduced and ordered
  - sometimes called reduced ordered BDDs (ROBDDs)
  - actually directed acyclic graphs (DAGs), not trees
  - compact, canonical representation for Boolean functions

- Variable ordering
  - a BDD assumes a fixed total ordering over its set of Boolean variables
  - e.g. $x_1 < x_2 < x_3$
  - along any path through the BDD, variables appear at most once each and always in the correct order

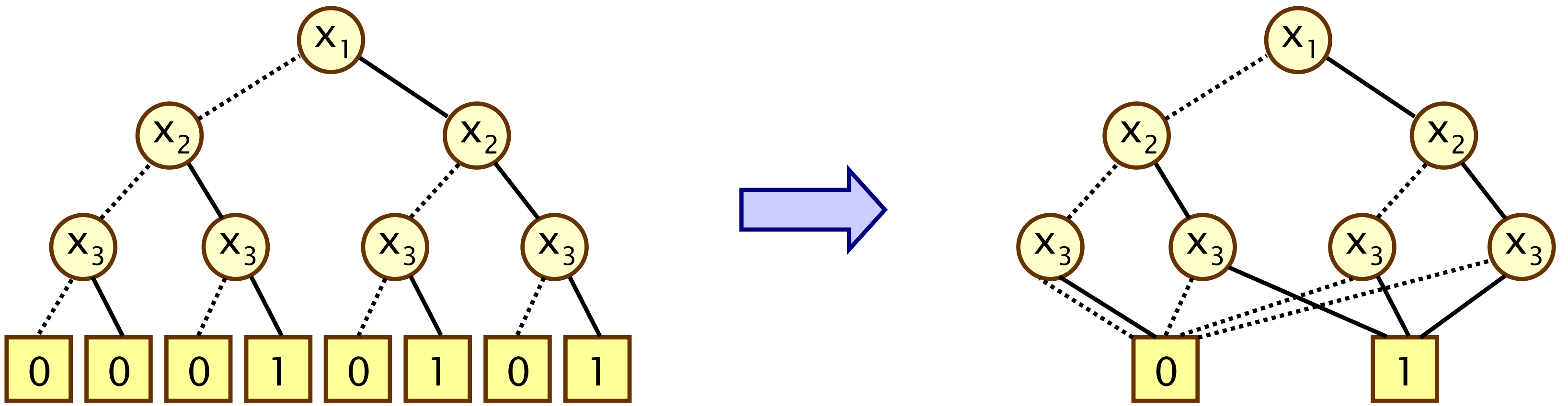


BDD reduction rule 1

- Rule 1: Merge identical terminal nodes

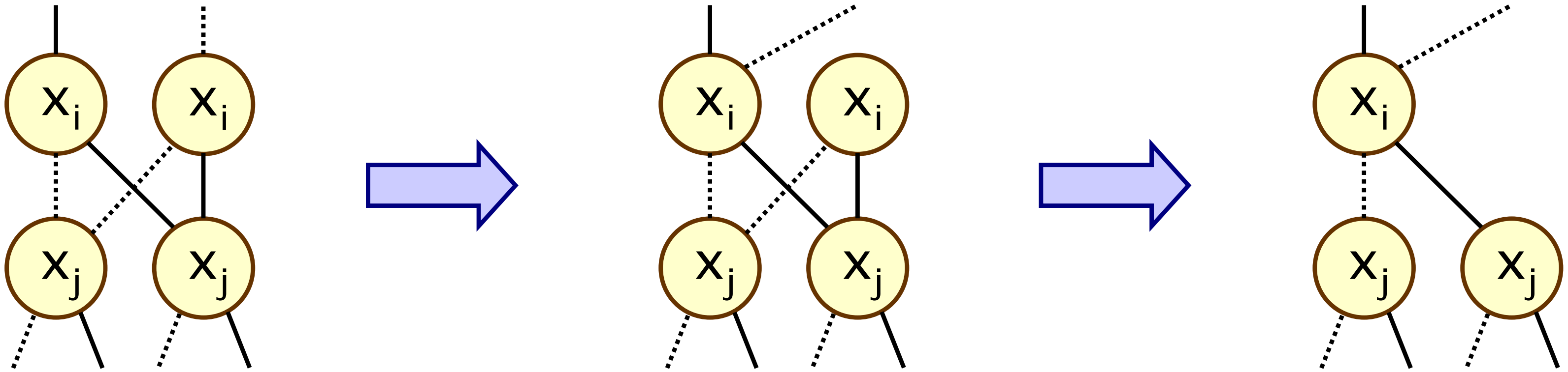


- Example:

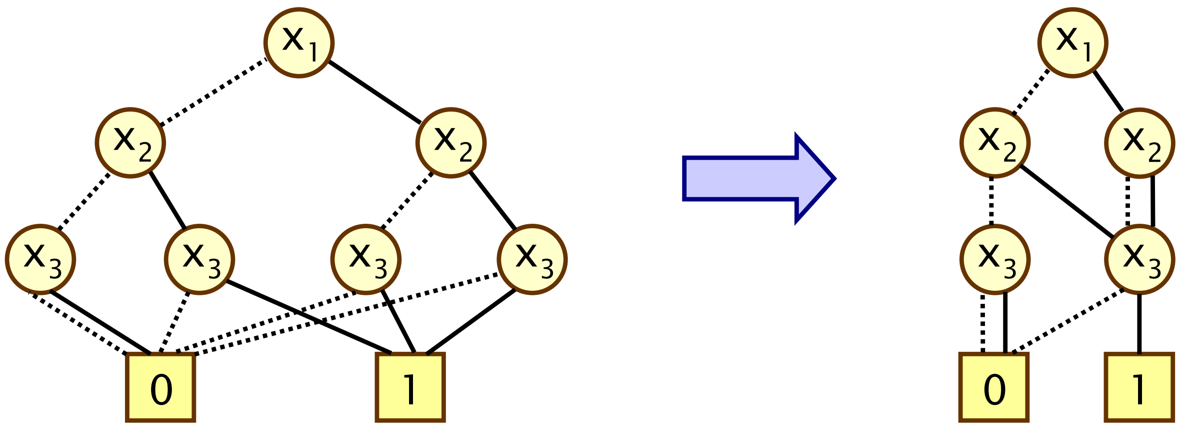


BDD reduction rule 2

- Rule 2: Merge isomorphic nodes, redirect incoming nodes

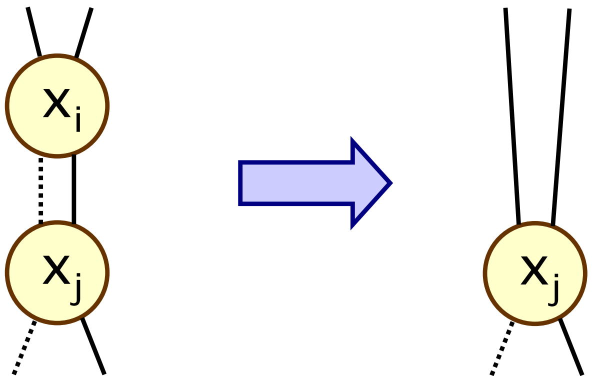


- Example:

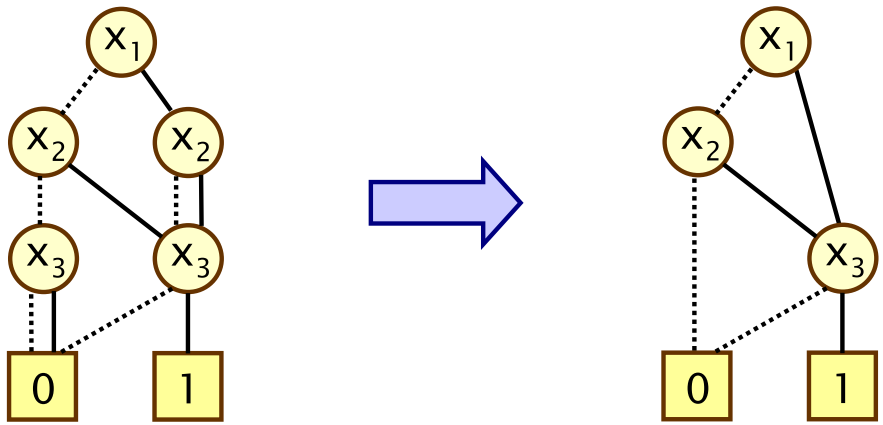


BDD reduction rule 3

- Rule 3: Remove redundant nodes (with identical children)

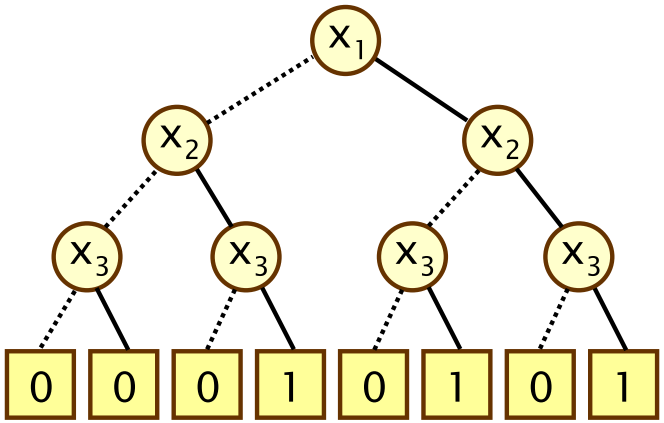


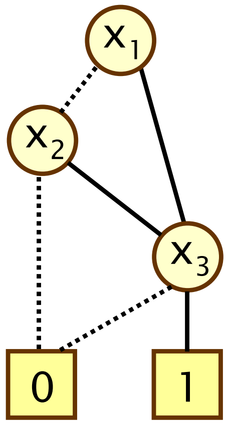
- Example:



Canonicity

- BDDs are a canonical representation for Boolean functions
  - two Boolean functions are equivalent if and only if the BDDs which represent them are isomorphic
  - uniqueness relies on: reduced BDDs, fixed variable ordered





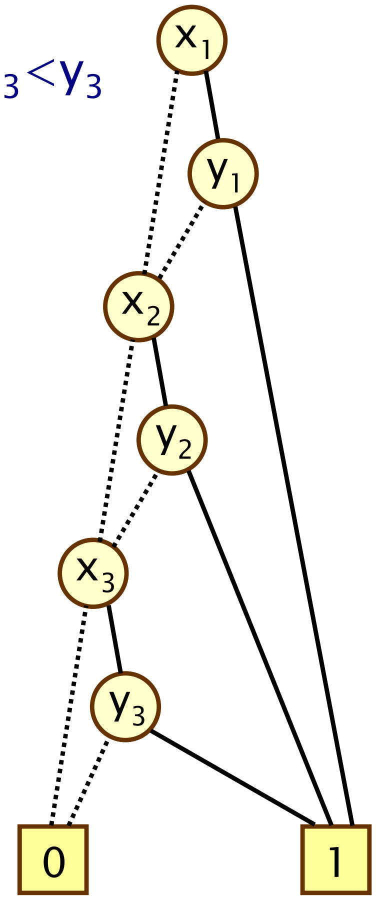
- Important implications for implementation efficiency
  - can be tested in linear (or even constant) time

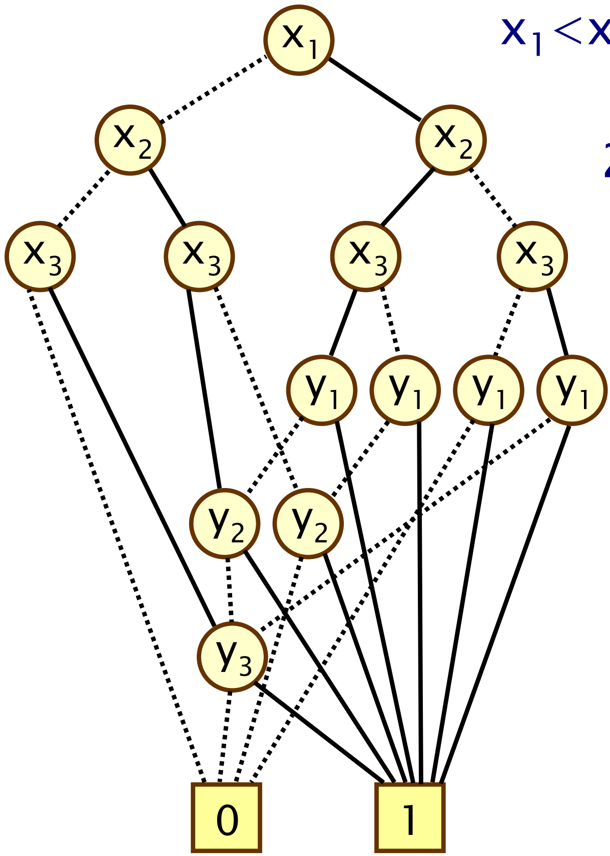
BDD variable ordering

- BDD size can be very sensitive to the variable ordering
  - example: $f = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee (x_3 \wedge y_3)$

$x_1<y_1<x_2<y_2< x_3<y_3$

2n+2 nodes





$x_1<x_2<x_3<y_1< y_2<y_3$

2^{n+1} nodes

BDDs to represent sets of states

- Consider a state space S and some subset $S' \subseteq S$

- We can represent S' by its characteristic function $\chi_{S'}$
  - $\chi_{S'} : S \rightarrow \{0,1\}$ where $\chi_{S'}(s) = 1$ if and only if $s \in S'$

- Assume we have an encoding of S into n Boolean variables
  - this is always possible for a finite set S
  - e.g. enumerate the elements of S and use a binary encoding
  - (note: there may be more efficient encodings though)

- So $\chi_{S'}$ can be seen as a function $\chi_{S'}(x_1,\ldots x_n) : \{0,1\}^n \rightarrow \{0,1\}$
  - which is simply a Boolean function
  - which can therefore be represented as a BDD

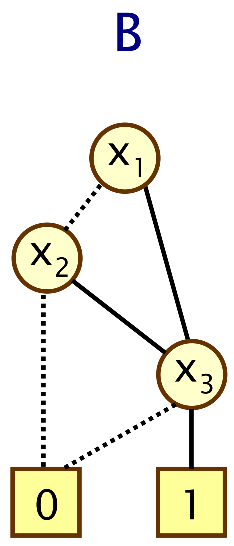
BDD and sets of states – Example

- State space S: {0, 1, 2, 3, 4, 5, 6, 7}
- Encoding of S: {000, 001, 010, 011, 100, 101, 110, 111}
- Subset S' ⊆ S: {3, 5, 7} → {011, 101, 111}

Truth table:

x_1	x_2	x_3	f_B
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

BDD:



BDDs and transition relations

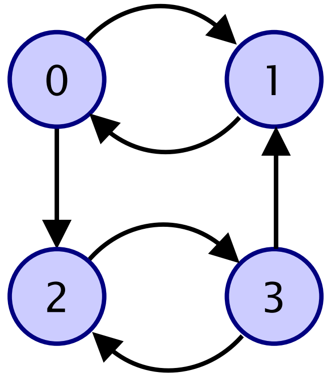
- Transition relations can also be represented by their characteristic function, but over pairs of states
  - relation: $R \subseteq S \times S$
  - characteristic function: $\chi_R : S \times S \to \{0,1\}$

- For an encoding of state space S into n Boolean variables
  - we have Boolean function $f_R(x_1,\ldots,x_n,y_1,\ldots,y_n) : \{0,1\}^{2n} \to \{0,1\}$
  - which can be represented by a BDD

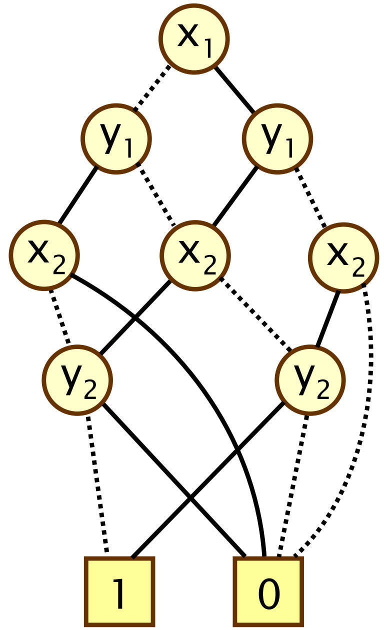
- Row and column variables
  - for efficiency reasons, we interleave the row variables $x_1,..,x_n$ and column variables $y_1,\ldots,y_n$
  - i.e. we use function $f_R(x_1,y_1,\ldots,x_n,y_n) : \{0,1\}^{2n} \to \{0,1\}$

BDDs and transition relations

- Example:
  - 4 states: 0, 1, 2, 3
  - Encoding: 0↦00, 1↦01, 2↦10, 3↦11



Transition	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$
(0,1)	0	0	0	1	0001
(0,2)	0	0	1	0	0100
(1,0)	0	1	0	0	0010
(2,3)	1	0	1	1	1101
(3,1)	1	1	0	1	1011
(3,2)	1	1	1	0	1110

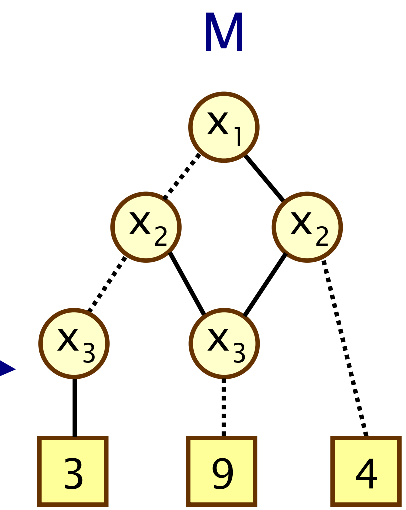


Multi-terminal binary decision diagrams

- Multi-terminal BDDs (MTBDDs), sometimes called ADDs
  - extension of BDDs to represent real-valued functions
  - like BDDs, an MTBDD M is associated with n Boolean variables
  - MTBDD M represents a function $f_M(x_1,\ldots,x_n) : \{0,1\}^n \rightarrow \mathbb{R}$

For clarity, we omit the zero terminal node and any incoming edges

e.g.



x_1	x_2	x_3	f_M
0	0	0	0
0	0	1	3
0	1	0	9
0	1	1	0
1	0	0	4
1	0	1	4
1	1	0	9
1	1	1	0

MTBDDs to represent vectors

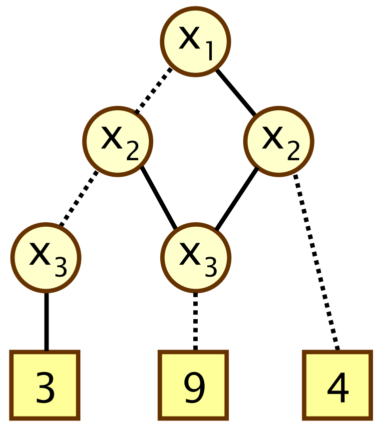
- In the same way that BDDs can represent sets of states…
  - MTBDDs can represent real-valued vectors over states S
  - e.g. a vector of probabilities Prob(s, ψ) for each state $s \in S$
  - assume we have an encoding of S into n Boolean variables
  - then vector $\underline{v} : S \to \mathbb{R}$ is a function $f_v(x_1,\ldots,x_n) : \{0,1\}^n \to \mathbb{R}$

Vector $\underline{v}$

[0,3,9,0,4,4,9,0]

x_1	x_2	x_3	i	f_v
0	0	0	0	0
0	0	1	1	3
0	1	0	2	9
0	1	1	3	0
1	0	0	4	4
1	0	1	5	4
1	1	0	6	9
1	1	1	7	0

MTBDD v



MTBDDs to represent matrices

- MTBDDs can be used to represent real-valued matrices indexed over a set of states S
  - e.g. the transition probability/rate matrix of a DTMC/CTMC

- For an encoding of state space S into n Boolean variables
  - a vector $\underline{v} : S \to \mathbb{R}$ is a function $f_v(x_1,\ldots,x_n) : \{0,1\}^n \to \mathbb{R}$
  - a matrix **M** maps pairs of states to reals i.e. $\mathbf{M} : S \times S \to \mathbb{R}$
  - this becomes: $f_M(x_1,\ldots,x_n,y_1,\ldots,y_n) : \{0,1\}^{2n} \to \mathbb{R}$

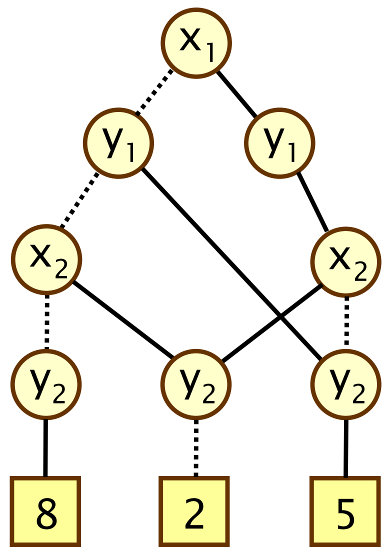
- Row and column variables
  - for efficiency reasons, we interleave the row variables $x_1,..,x_n$ and column variables $y_1,\ldots,y_n$
  - i.e. we use function $f_M(x_1,y_1,\ldots,x_n,y_n) : \{0,1\}^{2n} \to \mathbb{R}$

Matrices and MTBDDs – Example

Matrix **M**

$$\begin{bmatrix} 0 & 8 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

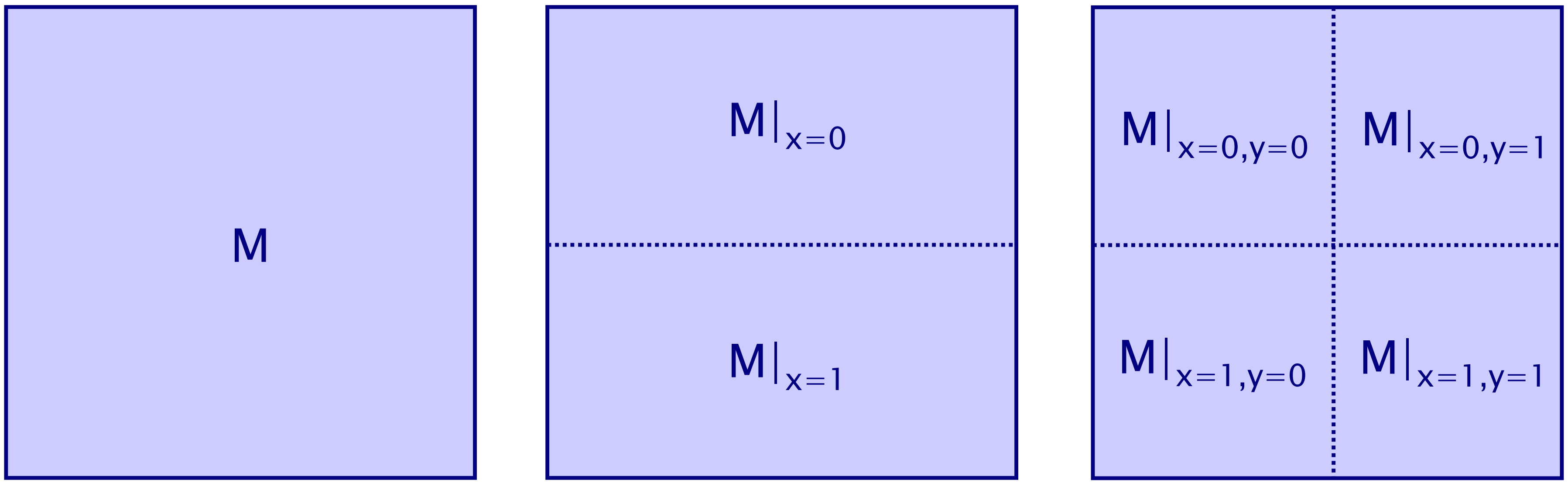
MTBDD M



Entry in M	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$	f_M
(0,1) = 8	0	0	0	1	0001	8
(1,0) = 2	0	1	0	0	0010	2
(0,3) = 5	0	0	1	1	0101	5
(1,3) = 5	0	1	1	1	0111	5
(2,3) = 5	1	0	1	1	1101	5
(3,2) = 2	1	1	1	0	1110	2

Matrices and MTBDDs – Recursion

- Descending one level in the MTBDD (i.e. setting $x_i=b$)
  - splits the matrix represented by the MTBDD in half
  - row variables (x_i) give horizontal split
  - column variables (y_i) give vertical split

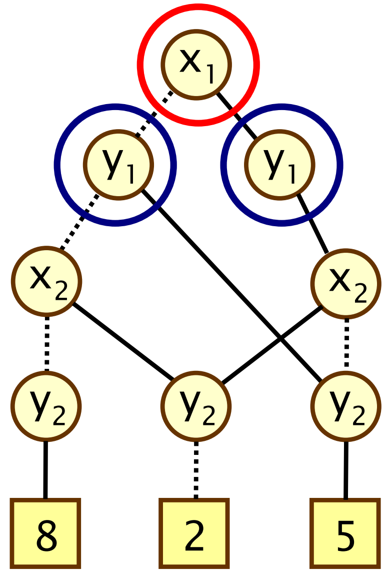


Matrices and MTBDDs – Recursion

Matrix **M**

$$\left[\begin{array}{cc|cc} 0 & 8 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ \hline 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

MTBDD M



Entry in M	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$	f_M
(0,1) = 8	0	0	0	1	0001	8
(1,0) = 2	0	1	0	0	0010	2
(0,3) = 5	0	0	1	1	0101	5
(1,3) = 5	0	1	1	1	0111	5
(2,3) = 5	1	0	1	1	1101	5
(3,2) = 2	1	1	1	0	1110	2

Matrices and MTBDDs – Regularity

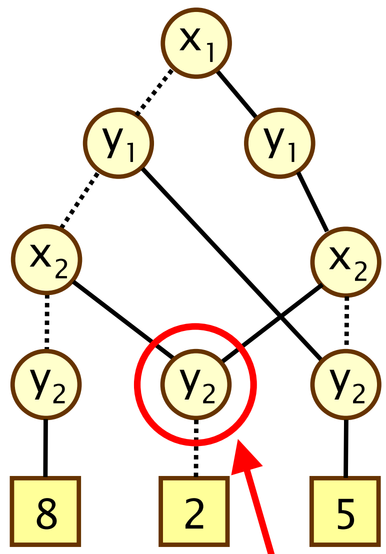
Matrix **M**

$$\begin{bmatrix} 0 & 8 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Repeated submatrices

Entry in M	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$	f_M
(0,1) = 8	0	0	0	1	0001	8
(1,0) = 2	0	1	0	0	0010	2
(0,3) = 5	0	0	1	1	0101	5
(1,3) = 5	0	1	1	1	0111	5
(2,3) = 5	1	0	1	1	1101	5
(3,2) = 2	1	1	1	0	1110	2

MTBDD M



Shared MTBDD node

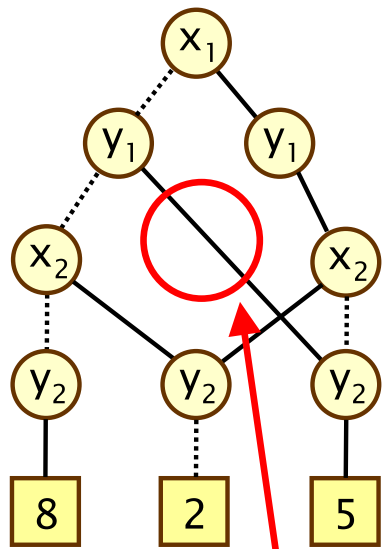
Matrices and MTBDDs – Regularity

Matrix **M**
$$\begin{bmatrix} 0 & 8 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Identical adjacent submatrices

Entry in M	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$	f_M
(0,1) = 8	0	0	0	1	0001	8
(1,0) = 2	0	1	0	0	0010	2
(0,3) = 5	0	0	1	1	0101	5
(1,3) = 5	0	1	1	1	0111	5
(2,3) = 5	1	0	1	1	1101	5
(3,2) = 2	1	1	1	0	1110	2

MTBDD M



MTBDD node removed

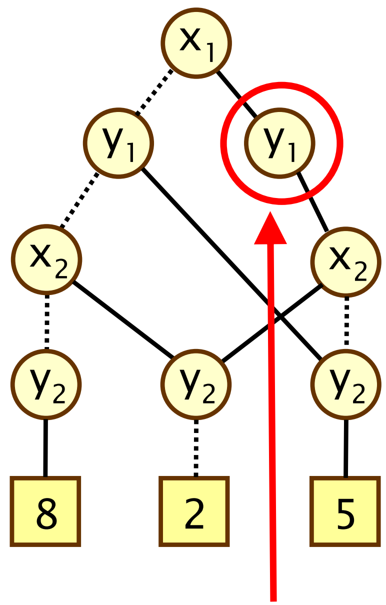
Matrices and MTBDDs – Sparseness

Matrix **M**

$$\begin{bmatrix} 0 & 8 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Blocks of zeros

MTBDD M



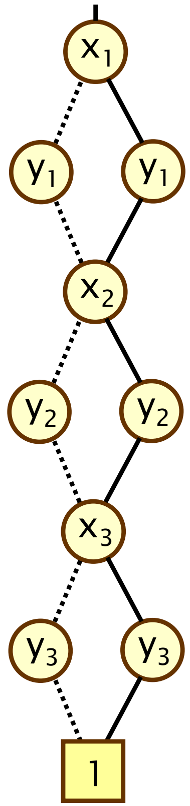
Edge goes straight to zero node

Entry in M	x_1	x_2	y_1	y_2	$x_1y_1x_2y_2$	f_M
(0,1) = 8	0	0	0	1	0001	8
(1,0) = 2	0	1	0	0	0010	2
(0,3) = 5	0	0	1	1	0101	5
(1,3) = 5	0	1	1	1	0111	5
(2,3) = 5	1	0	1	1	1101	5
(3,2) = 2	1	1	1	0	1110	2

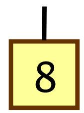
Matrices and MTBDDs – Compactness

- Some simple matrices have extremely compact representations as MTBDDs
  - e.g. the identify matrix or a constant matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 8 & 8 & 8 & 8 & \cdots \\ 8 & 8 & 8 & 8 & \cdots \\ 8 & 8 & 8 & 8 & \cdots \\ 8 & 8 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$



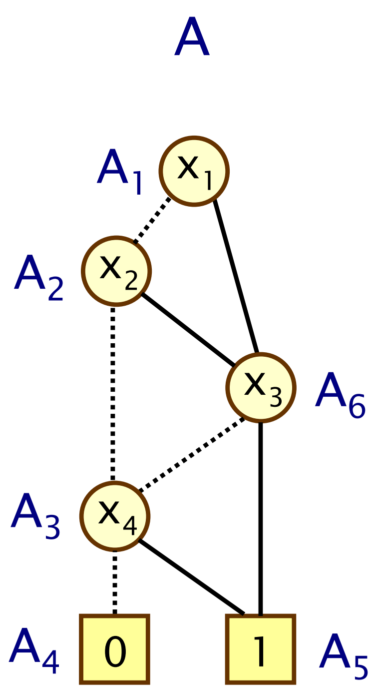
Manipulating BDDs

- Need efficient ways to manipulate Boolean functions
  - while they are represented as BDDs
  - i.e. algorithms which are applied directly to the BDDs
- Basic operations on Boolean functions:
  - negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), etc.
  - can all be applied directly to BDDs
- Key operation on BDDs: Apply(op, A, B)
  - where A and B are BDDs and op is a binary operator over Boolean values, e.g. $\wedge$, $\vee$, etc.
  - Apply(op, A, B) returns the BDD representing function f_A op f_B
  - often just use infix notation, e.g. Apply($\wedge$, A, B) = A $\wedge$ B
  - efficient algorithm: recursive depth-first traversal of A and B
  - complexity (and size of result) is O($|A| \cdot |B|$)
    - where $|C|$ denotes size of BDD C

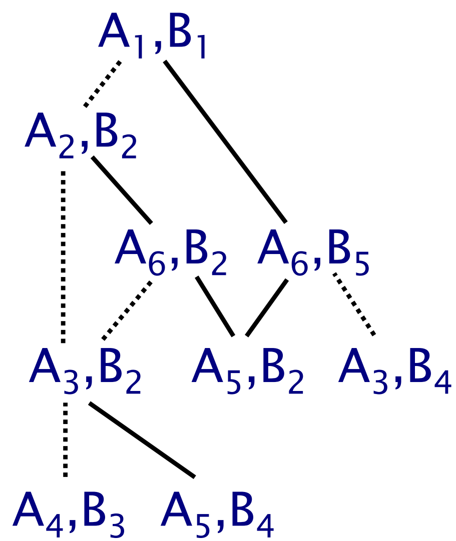
Apply – Example

- Example: Apply(∨, A, B)

Argument BDDs, with node labels:

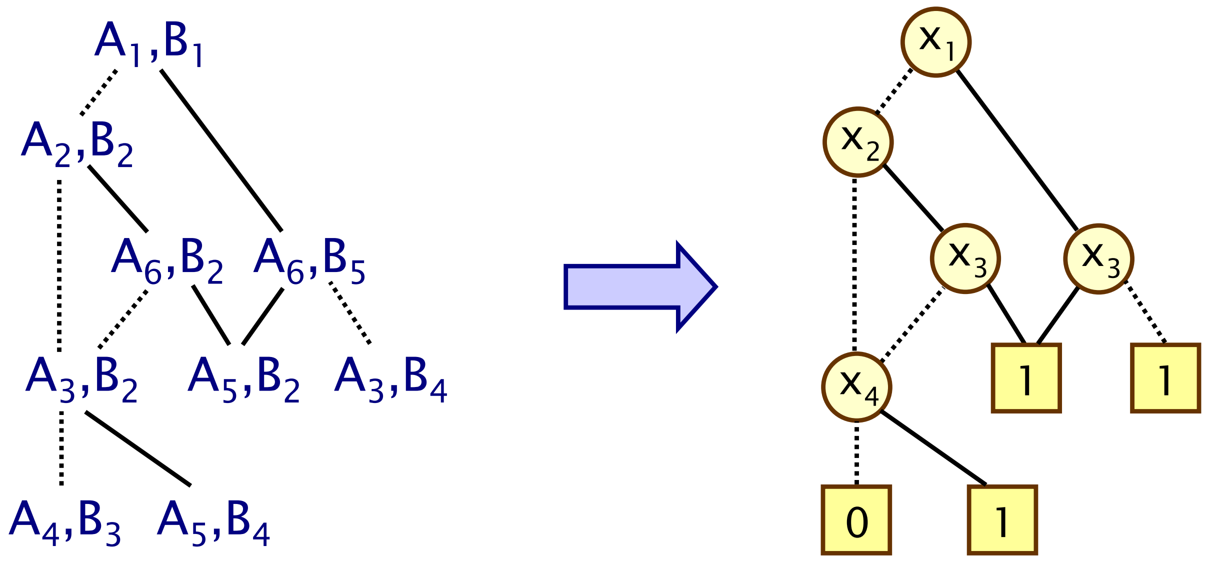


Recursive calls to Apply:



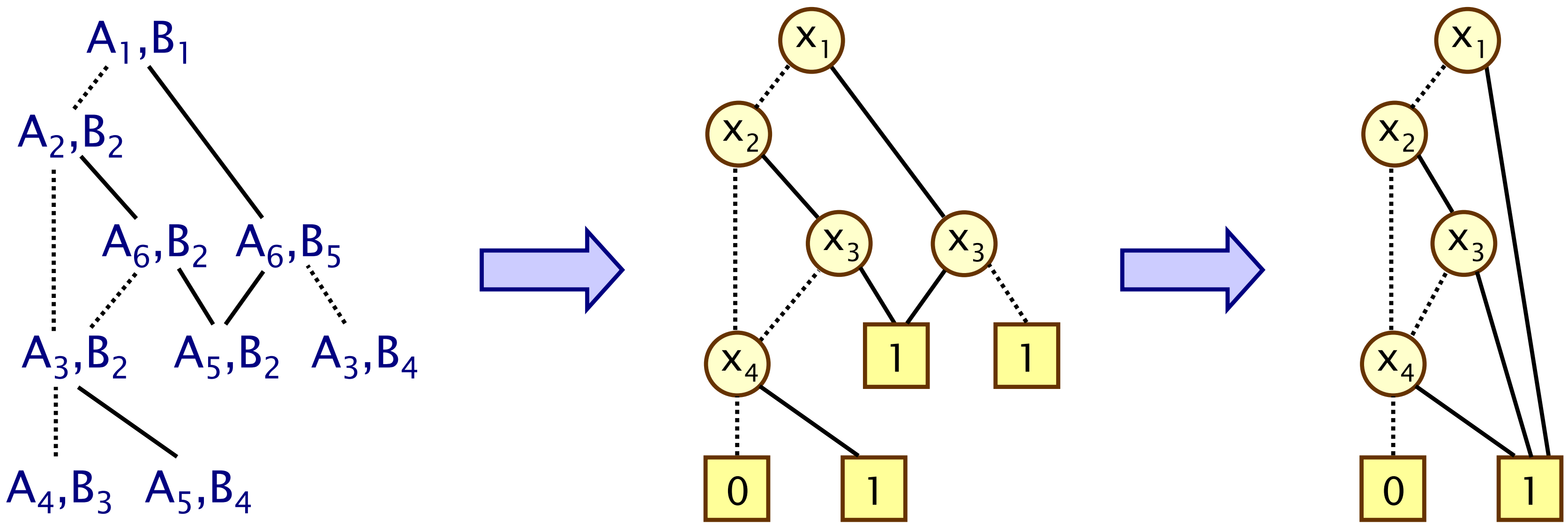
Apply – Example

- Example: Apply(∨, A, B)
  - recursive call structure implicitly defines resulting BDD



Apply – Example

- Example: Apply(∨, A, B)
  - but the resulting BDD needs to be reduced
  - in fact, we can do this as part of the recursive Apply operation, implementing reduction rules bottom-up



Implementation of BDDs

- Store all BDDs currently in use as one multi-rooted BDD
  - no duplicate BDD subtrees, even across multiple BDDs
  - every time a new node is created, check for existence first
  - sometimes called the "unique table"
  - implemented as set of hash tables, one per Boolean variable
  - need: node referencing/dereferencing, garbage collection
- Efficiency implications
  - very significant memory savings
  - trivial checking of BDD equality (pointer comparison)
- Caching of BDD operation results for reuse
  - store result of every BDD operation (memory dependent)
  - applied at every step of recursive BDD operations
  - relies on fast check for BDD equality

Operations with BDDs

- Operations on sets of states easy with BDDs
  - set union: $A \cup B$, in BDDs: $A \vee B$
  - set intersection: $A \cap B$, in BDDs: $A \wedge B$
  - set complement: $S \setminus A$, in BDDs: $\neg A$

- Graph-based algorithms (e.g. reachability)
  - need forwards or backwards image operator
    - i.e. computation of all successors/predecessors of a state
    - again, easy with BDD operations (conjunction, quantification)
  - other ingredients
    - set operations (see above)
    - equality of state sets (fixpoint termination) – equality of BDDs

Operations on MTBDDs

- The BDD operation Apply extends easily to MTBDDs
- For MTBDDs A, B and binary operation op over the reals:
  - Apply(op, A, B) returns the MTBDD representing f_A op f_B
  - examples for op: $+$, $-$, $\times$, min, max, …
  - often just use infix notation, e.g. Apply($+$, A, B) $=$ A $+$ B
- BDDs are just an instance of MTBDDs
  - in this case, can use Boolean ops too, e.g. Apply($\vee$, A, B)
- The recursive algorithm for implementing Apply on BDDs
  - can be reused for Apply on MTBDDs

Some other MTBDD operations

- Threshold(A, ~, c)
  - for MTBDD A, relational operator op and bound $c \in \mathbb{R}$
  - converts MTBDD to BDD based on threshold ~c
  - i.e. builds BDD representing function $f_A \sim c$
  - e.g. computing the underlying transition relation from the probability matrix of a DTMC: R = Threshold(P, >, 0)

- Abstract(op, $\{x_1,\ldots,x_n\}$, A)
  - for MTBDD A, variables $\{x_1,\ldots,x_n\}$ and commutative/associative binary operator over reals op
  - analogue of existential/universal quantification for BDDs
  - e.g. Abstract(+, {x}, A) constructs the MTBDD representing the function $f_{A|x=0} + f_{A|x=1}$
  - e.g. for BDD A: $\exists(x_1,..,x_n).A \equiv$ Abstract($\vee$, $\{x_1,\ldots,x_n\}$, A)

MTBDD matrix/vector operations

- Pointwise addition/multiplication and scalar multiplication
  - can be implemented with the Apply operator
  - Matrices: **A** + **B**, MTBDDs: Apply(+, A, B)

- Matrix-matrix multiplication **A·B**
  - can be expressed recursively based on 4-way matrix splits

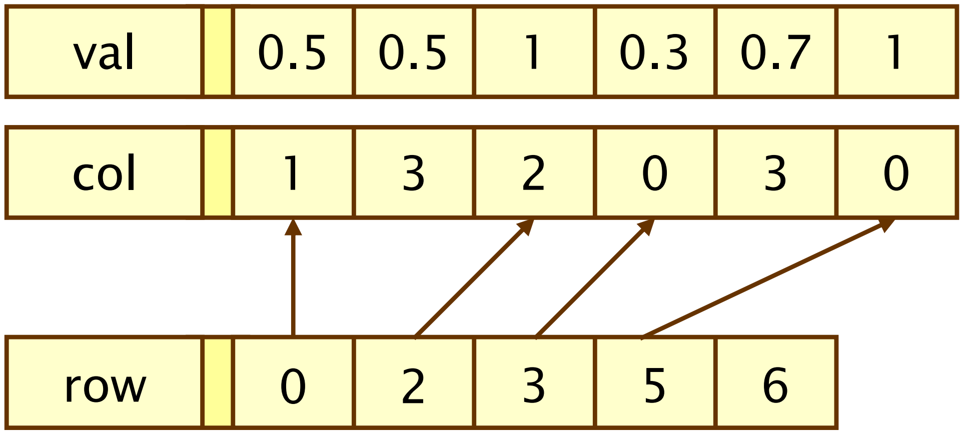
$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} \qquad \mathbf{A}_1 = \mathbf{B}_1 \cdot \mathbf{C}_1 + \mathbf{B}_2 \cdot \mathbf{C}_3, \text{ etc.}$$

  - which forms the basis of an MTBDD implementation
  - various optimisations are possible

- Matrix-matrix multiplication **A**·$\underline{\text{v}}$ is done in similar fashion

Sparse matrices

- Explicit data structure for matrices with many zero entries
  - assume a matrix **P** of size n × n with nnz non-zero elements
  - store three arrays: val and col (of size nnz) and row (of size n)
  - for each matrix entry (r,c)=v, c and v are stored in col/val
  - entries are grouped by row, with pointers stored in row
  - also possible to group by column



$$\mathbf{P} = \begin{bmatrix} \cdot & 0.5 & \cdot & 0.5 \\ \cdot & \cdot & 1 & \cdot \\ 0.3 & \cdot & \cdot & 0.7 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

Sparse matrices

- Advantages
  - compact storage (proportional to number of non-zero entries)
  - fast access to matrix entries
  - especially if usually need an entire row at once
  - (which is the case for e.g. matrix-vector multiplication)

- Disadvantage
  - less effficient to manipulate (i.e. add/delete matrix entries)

- Storage requirements
  - for a matrix of size $n \times n$ with nnz non-zero elements
  - assume reals are 8 byte doubles, indices are 4 byte integers
  - we need $8 \cdot nnz + 4 \cdot nnz + 4 \cdot n = 12 \cdot nnz + 4 \cdot n$ bytes

Sparse matrices vs. MTBDDs

- Storage requirements
  - MTBDDs: each node is 20 bytes
  - sparse matrices: 12·nnz+4·n bytes (n states, nnz transitions)
- Case study: Kanban manufacturing system, N jobs
  - store transition rate matrix **R** of the corresponding CTMCs

N	States (n)	Transitions (nnz)	MTBDD (KB)	Sparse matrix (KB)
3	58,400	446,400	48	5,459
4	454,475	3,979,850	96	48,414
5	2,546,432	24,460,016	123	296,588
6	11,261,376	115,708,992	154	1,399,955
7	41,644,800	450,455,040	186	5,441,445
8	133,865,325	1,507,898,700	287	13,193,599

Implementation in PRISM

- PRISM is a symbolic probabilistic model checker
  - the key underlying data structures are MTBDDs (and BDDs)

- In fact, has multiple numerical computation engines

  - MTBDDs: storage/analysis of very large models (given structure/regularity), numerical computation can blow up

  - Sparse matrices: fastest solution for smaller models ($<10^6$ states), prohibitive memory consumption for larger models

  - Hybrid: combine MTBDD storage with explicit storage, ten-fold increase in analysable model size ($\sim 10^7$ states)

Summing up…

- Implementation of probabilistic model checking
  - graph-based algorithms, e.g. reachability, precomputation
  - manipulation of sets of states, transition relations
  - iterative numerical computation
  - key operation: matrix-vector multiplication
- Binary decision diagrams (BDDs)
  - representation for Boolean functions
  - efficient storage/manipulation of sets, transition relations
- Multi-terminal BDDs (MTBDDs)
  - extension of BDDs to real-valued functions
  - efficient storage/manipulation of real-valued vectors, matrices (assuming structure and regularity)
  - can be much more compact than (explicit) sparse matrices